Mathematics - Course 221

APPLICATION OF THE INTEGRAL AS AN INFINITE SUM

I Area Under a Curve

Suppose the x-interval from x = a to x = b is *partitioned* into n subintervals of width

$$\Delta x = \frac{b - a}{n}$$

Then the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is approximately the same as the area of the n inscribed rectangles corresponding to these n subintervals, as shown in Figure 1.

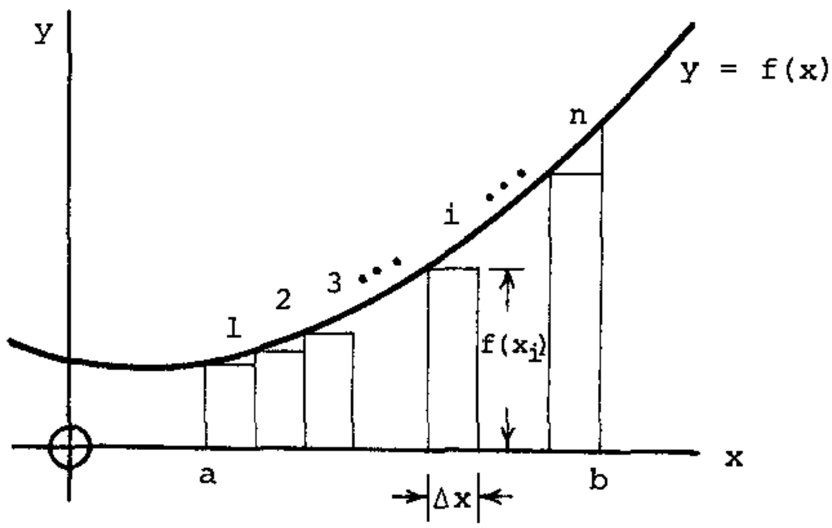


Figure 1

In Figure 1, if x_i is the x-value at the left boundary of the i th rectangle, then $f(x_i)$ is the height of the i th rectangle, and the total area A_r of the n rectangles is

$$A_r = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x$$

This sum can be abbreviated using *sigma notation* as follows:

$$A_r = \sum_{i=1}^{n} f(x_i)\Delta x$$

The RHS of this equation is read "the sum of $f(x_i)\Delta x$ as i takes all (integral) values from 1 to n inclusive".

As can be seen from Figure 2, which shows the interval of Figure 1 partitioned into n = 4, 8, 16 subintervals, the larger the number n of rectangles, the closer A_r is to the true area A under the curve. In fact,

$$A = \lim_{n\to\infty} A_r$$

ie,
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

But, from lesson 221.30-1,

$$A = \int_a^b f(x)\,dx$$

$$\therefore \quad \boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, \quad \Delta x = \frac{b - a}{n}}$$

ie, the definite integral can be regarded as the sum of an infinite number of infinitely narrow rectangles. Note that as i ranges from 1 to n in the sum, x ranges from a to b in the integral. In fact, the integral sign was originally introduced as a stretched "S" standing for "sum".

This notion of the definite integral as an infinite sum of vanishingly small increments is extremely useful in a wide range of applications, a few of which will be discussed in this lesson. The above result is often referred to as *The Fundamental Theorem.*

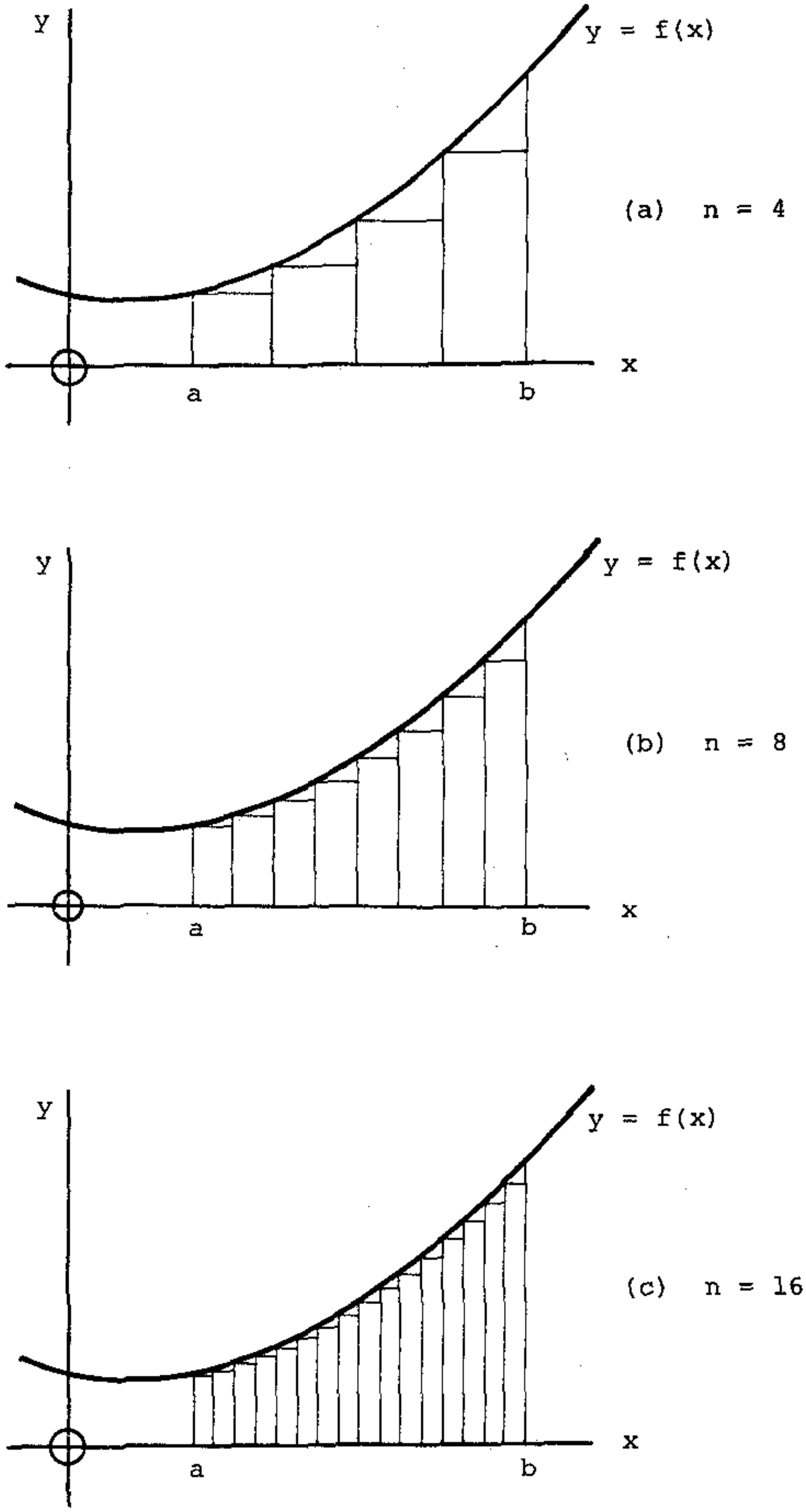


Figure 2

Example 1

Find the area enclosed between the curves of $y = x^2$ and $x = y^2$.

Solution

A large, clear diagram showing a representative rectangular slice of the required area is mandatory in the solution to such problems.

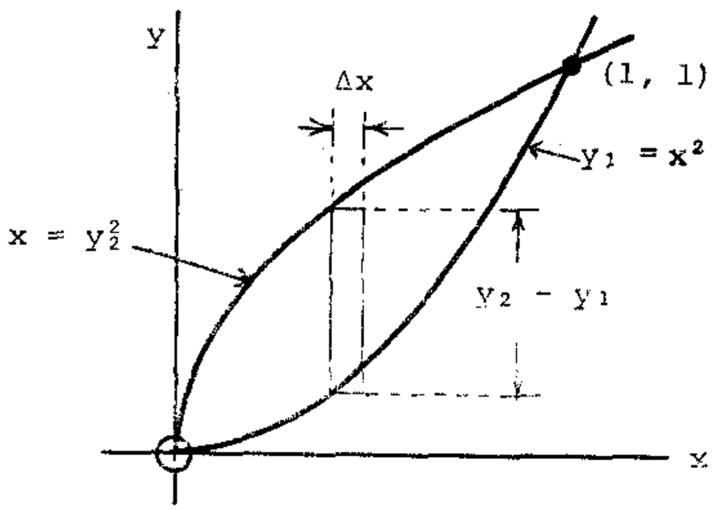


Figure 3

Clearly, from Figure 3, the limits of integration will be the x-values where the two curves intersect. These values are found by solving the two equations $x = y_2^2$ and $y_1 = x^2$:

$$y_1 = y_2 \quad \Rightarrow \quad x^2 = \sqrt{x}$$

$$\text{ie, } x^4 = x$$

$$\text{ie, } x^4 - x = 0$$

$$\text{ie, } x(x^3 - 1) = 0$$

$$\therefore\ x = 0 \text{ or } x = 1$$

$\therefore$ the curves intersect at (0, 0) and (1, 1).

The required area, $A = \lim_{n\to\infty} \Sigma (y_2 - y_1)\Delta x$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (\sqrt{x_i} - x_i^2)\Delta x, \quad \Delta x = \frac{1 - 0}{n}$$

$$= \int_{x=0}^{1} (\sqrt{x} - x^2)\,dx \quad \text{(by Fundamental Theorem)}$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} - (0 - 0)$$

$$= \frac{1}{3} \text{ square units}$$

To illustrate the versatility of this technique the same area will be calculated again, this time using horizontal slices as shown in Figure 4:

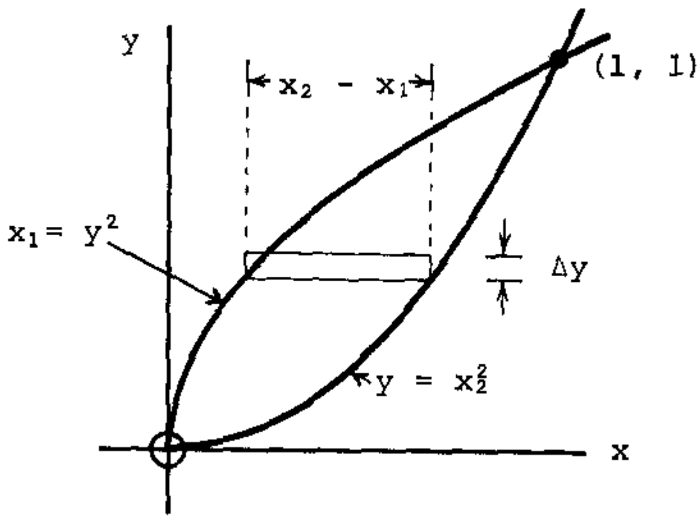


Figure 4

$$A = \text{sum of horizontal slices}$$

$$= \lim_{n\to\infty} \Sigma(x_2 - x_1)\Delta y$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (\sqrt{y_i} - y_i^2)\Delta y, \quad \Delta y = \frac{1-0}{n}$$

$$= \int_{y=0}^{1} (\sqrt{y} - y^2)\, dy$$

$$= \left[\frac{2}{3} y^{3/2} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$= \frac{1}{3} \text{ square units as before.}$$

Example 2

Find the area bounded by the parabola $y = x^2 - 8x + 7$, the x-axis, and the lines $x = 2$ and $x = 6$.

Solution

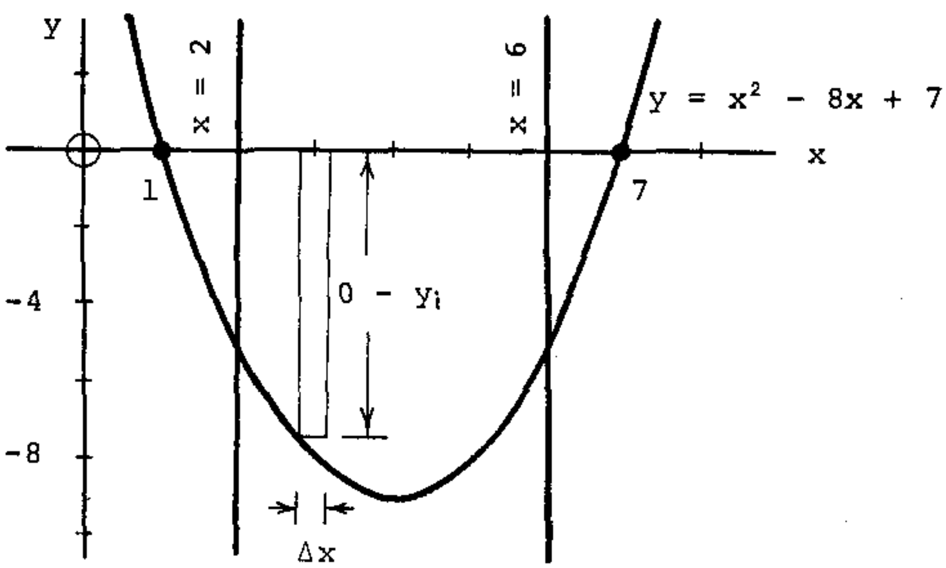


Figure 5

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (0 - y_i)\Delta x \qquad \text{(see Figure 5)}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (-x_i{}^2 + 8x_i - 7)\Delta x, \quad \Delta x = \frac{6-2}{n}$$

$$= \int_{x=2}^{6} (-x^2 + 8x - 7)dx \quad \text{(by Fundamental Theorem)}$$

$$= \left[-\frac{x^3}{3} + 4x^2 - 7x\right]_2^6$$

$$= \left[-\frac{6^3}{3} + 4(6)^2 - 7(6)\right] - \left[-\frac{2^3}{3} + 4(2)^2 - 7(2)\right]$$

$$= 30 - \left[-\frac{2}{3}\right]$$

$$= \underline{\underline{30\tfrac{2}{3} \text{ square units.}}}$$

II Calculating Averages

Recall that calculating an average involves dividing a sum of parts by the number of parts (cf lesson 321.30-1).

Example 3

Find the average of the following lengths: 2.10 m, 95 cm, 123 cm, 4.20 m.

Solution

$$\text{Average length} = \frac{2.10 + 0.95 + 1.23 + 4.20}{4}$$

$$= \underline{\underline{2.12 \text{ m}}}$$

It is often necessary to calculate a weighted average, as in Example 4.

Example 4

During a two-year period, the annual inflation rate averaged 8.4% during the first year. During the second year, the rate averaged 9.2% over the first quarter, 9.8% over the second quarter, and 10.2% over the second half. Calculate the mean rate over the two-year period.

Solution

The variation of the inflation rate R is shown in Figure 6.

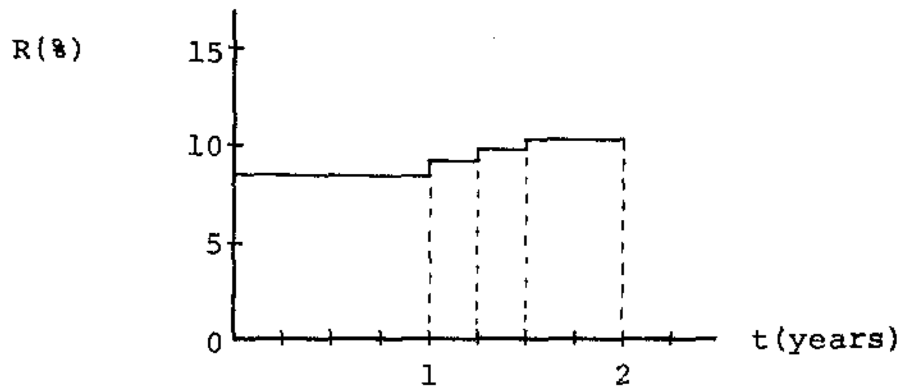


Figure 6

$$\text{Average Inflation Rate } \bar{R} = \frac{1.00 \times 8.4 + 0.25 \times 9.2 + 0.25 \times 9.8 + 0.50 \times 10.2}{2.00}$$

$$= \underline{\underline{9.1\%}}$$

Note that the above calculation is equivalent to dividing the area under the graph of Figure 6 by the total time interval:

$$\bar{R} = \frac{\text{area under R - t graph}}{\text{total t-interval}}$$

This latter approach is useful in calculating averages of continuously varying quantities.

Example 5

The radiation dose rate from a certain radioactive source decays exponentially with time according to the formula,

$$D(t) = D_0 e^{-\lambda t},$$

where, D_0 = 500 mrem/h is the dose rate at t = 0, and
$\lambda \approx 7.6 \times 10^{-3}\ s^{-1}$ is the decay constant of the source activity.

Find (a) the average dose rate during the first 5 hours.

(b) the total dose received during the first 5 hours.

Solution

t (hours)	0	1	2	3	4	5
D(t) (mrem/h)	500	380	289	220	167	127

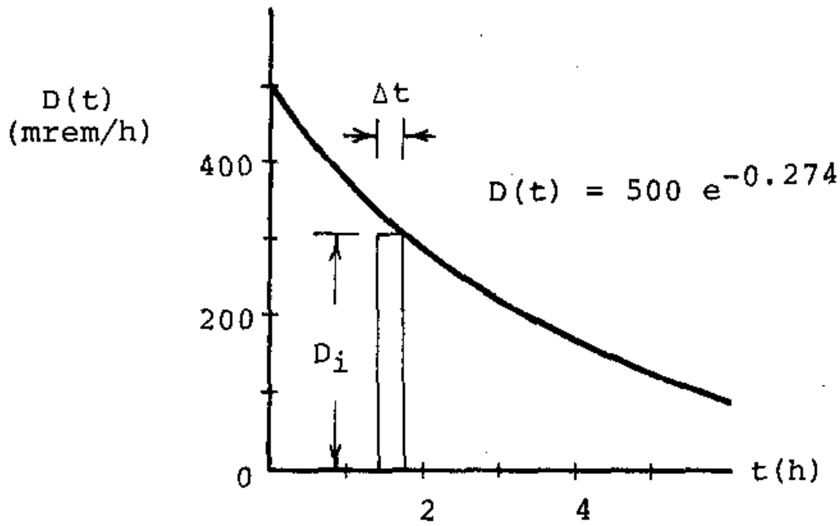


Figure 7

(a) Average dose rate $= \dfrac{\text{total dose received}}{\text{total time}}$

$$= \frac{\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} D_i \Delta t}{5}$$

$$= \frac{1}{5} \lim_{n\to\infty} \sum_{i=1}^{n} 500e^{-\lambda t_i} \Delta t$$

$$= -\frac{1}{5\lambda} \int_{t=0}^{5} 500e^{-\lambda t}(-\lambda)\,dt \quad \text{(by Fundamental Theorem)}$$

$$= \frac{100}{-\lambda} e^{-\lambda t}\Big|_0^5$$

$$= \frac{100}{-\lambda} (e^{-0.274 \times 5} - e^{\circ})$$

$$= \frac{100}{0.274} (1 - e^{-1.37})$$

$$= \underline{\underline{2.7 \times 10^2 \text{ mrem/h}}}$$

(b) Total dose $=$ average dose rate x 5

$$= 272 \times 5$$

$$= 1.4 \times 10^3 \text{ mrem}$$

$$= \underline{\underline{1.4 \text{ rem}}}$$

Example 6

The number N(t) of radioactive nuclei surviving to time t seconds is given by the formula

$$N(t) = N_0 e^{-\lambda t} ,$$

Where N_0 is the number of radioactive nuclei at $t = 0$ and λ is the decay constant.

Find the mean lifetime of a nucleus.

Solution

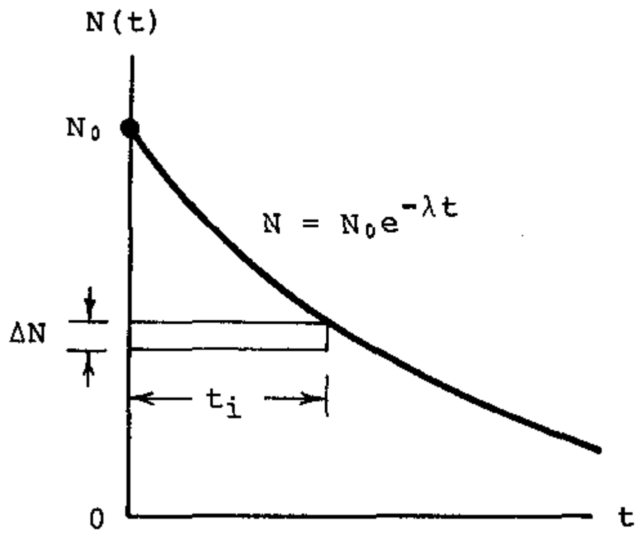


Figure 8

$$\text{Average nuclear lifetime} = \frac{\text{total lifetime of all nuclei}}{\text{total number of nuclei}}$$

$$= \frac{\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} t_i \Delta N}{N_0}$$

$$= \frac{1}{N_0} \int_{N=0}^{N_0} t\,dN$$

The integration techniques necessary to evaluate this integral have not been covered in this text. Nevertheless, the answer is that the mean nuclear lifetime, T, equals $1/\lambda$. Thus an alternative form of the given equation is

$$N(t) = N_0 e^{-t/T},$$

from which it is obvious that each time one mean nuclear lifetime passes, the number of surviving radioactive nuclei drops by a factor of e.

III Work

Recall from elementary mechanics that work W equals the product of force F times displacement s:

$$W = Fs$$

Example 7

The work done in lifting a 2 kg mass 5 m against gravity,

$$\begin{aligned} W &= F_g s \\ &= mgs \qquad \text{(weight } F_g = mg) \\ &= 2 \times 9.8 \times 5 \quad (g = 9.8\ m/s^2) \\ &= \underline{\underline{98\ J}} \end{aligned}$$

In Example 7, the force is constant. When the force is not constant, in general, the work must be calculated by integration.

Example 8

The force F required to stretch a spring varies as the spring extension x,

ie, $F = kx$, where k is the *spring constant*.

Find the amount of energy stored in a spring stretched 0.40 m, assuming $k = 2.0 \times 10^2\ Nm^{-1}$.

Solution

The extension force F is plotted versus extension x in Figure 9.

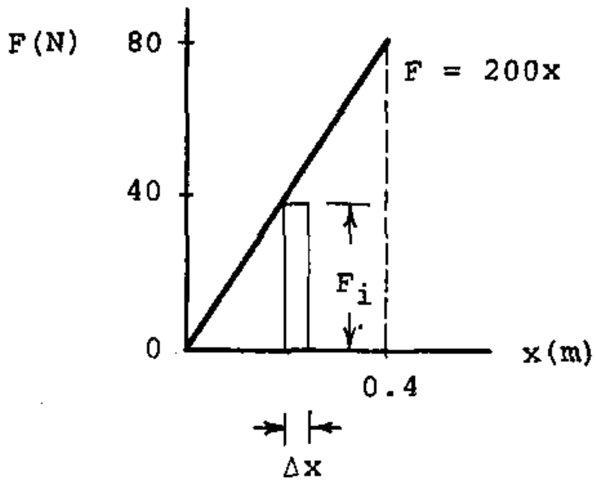


Figure 9

The amount of work represented by the area of the i th rectange in Figure 9 is

$$\Delta W_i = F_i \Delta x \quad \text{(Force } F_i \text{ acts through } \Delta x\text{)}$$

Total work W done in stretching the spring to x = 0.4 m equals total area under F - s graph from x = 0 to x = 0.4,

ie,
$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta W_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} F_i \Delta x$$

$$= \int_0^{0.4} F dx \qquad \text{(by Fundamental Theorem)}$$

$$= \int_0^{0.4} kx dx$$

$$= \frac{1}{2} kx^2 \Big|_0^{0.4}$$

$$= \frac{k}{2}\,(0.4^2 - 0^2)$$

$$= \frac{2.0 \times 10^2}{2}\,(0.16)$$

$$= \underline{\underline{16\ J}}$$

$\therefore$ 16 J of elastic potential energy is stored in the stretched spring.

IV Fluid Pressure

Recall from fluid mechanics that the pressure P exerted by a fluid of density ρ kg/m^3 at a depth of h meters is given by the formula.

$$P = \rho g h \quad \text{pascals,}$$

where $g = 9.8$ m/s^2 is the acceleration due to gravity.

Example 9

For the trapezoidal hydraulic dam of Figure 10 (dimensions in meters) find the

(a) total force exerted against the dam face

(b) average pressure exerted against the dam face

(c) center of pressure $(0, \bar{y})$ exerted against dam face, where

$$\bar{y} = \frac{M_x}{F_t}\ , \text{ and}$$

$M_x = \int yPdA$ is called the *moment of force about the x-axis*, and

$F_t = \int PdA$ is the total force against the dam,

ie, the same moment would exist if F_t were applied at $(0, \bar{y})$.

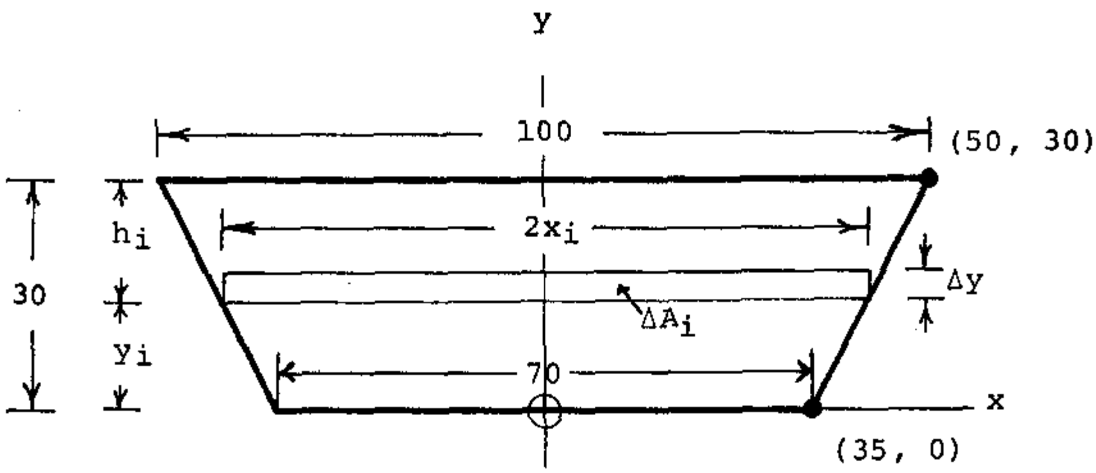


Figure 10

Solution

Total force F_t against dam equals sum of forces ΔF_i against rectangular increments,

$$\text{ie,}\quad F_t = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta F_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P_i \Delta A_i \qquad (\Delta F_i = P_i \Delta A_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (\rho g h_i)(2x_i \Delta y) \qquad (P_i = \rho g h_i;\ \Delta A_i = 2x_i \Delta y)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \rho g(30 - y_i)\ 2x_i \Delta y \qquad (h_i = 30 - y_i)$$

$$= \int_{y=0}^{30} \rho g(30 - y)\ 2x\,dy \qquad \text{(by Fundamental Theorem)}$$

In order to evaluate this integral, one must express x in terms of y. This is done by first finding the equation of the line representing the right boundary of the dam in Figure 10, using the two-point form:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \text{where } (x_1, y_1) = (35, 0)$$

$$\text{and } (x_2, y_2) = (50, 30)$$

ie, $\dfrac{y - 0}{x - 35} = \dfrac{30 - 0}{50 - 35}$

ie, $30(x - 35) = 15y$

ie, $x = \dfrac{y}{2} + 35$

Substituting for x in the above integral, one gets

$$F_t = \int_0^{30} \rho g(30 - y)\ 2(\tfrac{y}{2} + 35)dy$$

$$= \rho g\int_0^{30} (30 - y)(y + 70)dy$$

$$= \rho g\int_0^{30} (2100 - 40y - y^2)dy$$

$$= \rho g\left[2100y - 20y^2 - \frac{y^3}{3}\right]_0^{30}$$

$$= 10^3 \times 9.8 \left[2100 \times 30 - 20(30)^2 - \frac{30^3}{3}\right]$$

$$= \underline{\underline{3.5 \times 10^8 \text{ N}}}$$

(b) Average pressure on dam $\overline{P} = \dfrac{\text{total force on dam}}{\text{total area of dam}}$

$$= \frac{3.53 \times 10^8 \text{ N}}{\frac{1}{2}(100 + 70)30 \text{ m}^2}$$

$$= 1.4 \times 10^5 \text{ Pa}$$

$$= \underline{\underline{0.14 \text{ MPa}}}$$

<u>Note</u>: (1) Dam area was calculated using formula for area of a trapezoid,

$A = \frac{1}{2}(s_1 + s_2)h$, where s_1, s_2 are the lengths of the parallel sides, and h is the perpendicular distance between them.

(2) For comparison with above result for $\overline{P}$, the pressure at half-depth (15 m) is

$$\rho gh = 10^3 \times 9.8 \times 15$$
$$= 0.15 \text{ MPa}$$

$\overline{P}$ is slightly less than this because the dam widens towards the top, ie lower pressures are exerted on the wider slices. If the dam had vertical boundaries, $\overline{P}$ would equal the pressure at half depth.

(c) $\overline{y} = \frac{M_x}{F_t}$,

where $F_t = 3.53 \times 10^8$ Pa from (a), and

$$M_x = \int_{y=0}^{30} yPdA$$

$$= \int_0^{30} y\rho gh\ 2xdy$$

$$= \rho g\int_0^{30} y(30 - y)(y + 70)dy$$

$$= \rho g\int_0^{30} (2100y - 40y^2 - y^3)dy$$

$$= \rho g\left[1050y^2 - \frac{40}{3}y^3 - \frac{y^4}{4}\right]_0^{30}$$

$$= 10^3 \times 9.8\left\{1050(30)^2 - \frac{40}{3}(30)^3 - \frac{30^4}{4}\right]$$

$$= 3.75 \times 10^9 \text{ Nm}$$

$\therefore \quad \overline{y} = \frac{3.75 \times 10^9 \text{ Nm}}{3.53 \times 10^8 \text{ Nm}^{-2}}$

$$= \underline{\underline{11 \text{ m}}}$$

$\therefore$ centre of pressure against dam is at (0, 11) relative to axes of Figure 10.

V Other Applications of the Fundamental Theorem

Applications of the Fundamental Theorem are numerous in science and technology. A few more examples are listed below, but it is beyond the scope of this course to treat them in detail.

(1) Finding the volume of 3-dimensional entities by partitioning the volume into increments such as cubes, slices, or shells.

(2) Finding the center of mass of 3-dimensional entities of either homogeneous or variable densities.

(3) Finding the average neutron flux in reactor cores of various geometries.

(4) Finding moments of inertia of plane or 3-dimensional entities about an axis of rotation

(5) Finding surface area of 3-dimensional figures.

VI Familiar Instruments Which Integrate

By considering the integrating functions of certain familiar instruments, the trainee will consolidate his concept of integration, and better his chances of understanding the same function performed by other (perhaps less familiar) instruments.

(1) The watt-hour meter integrates the power P flowing to the consumer with respect to time, thus obtaining a record of electrical energy consumed, ie, it is effectively finding the area under the P-t curve (see Figure 11).

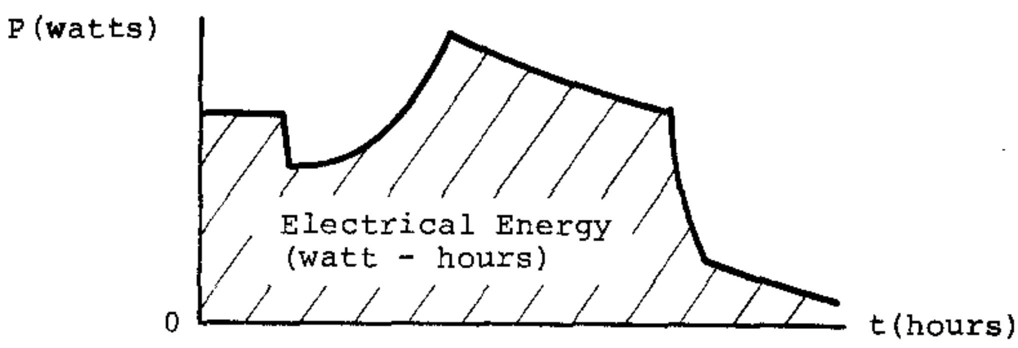


Figure 11

(2) A natural gas meter integrates the volumetric flow rate to the consumer, thus recording total volume consumed (see Figure 12).

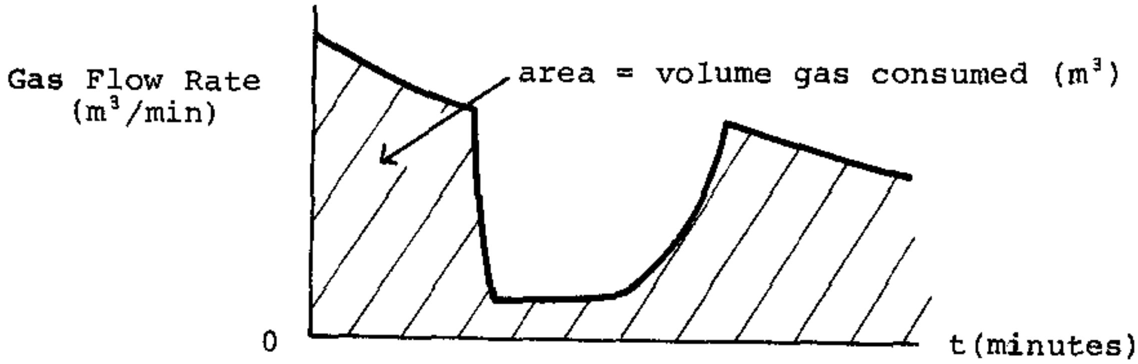


Figure 12

(3) The meter on a service station gasoline pump integrates flow rate of gasoline flowing into the consumer's gas tank, and records the total volume purchased, ie, it gives the area under the curve of Figure 13.

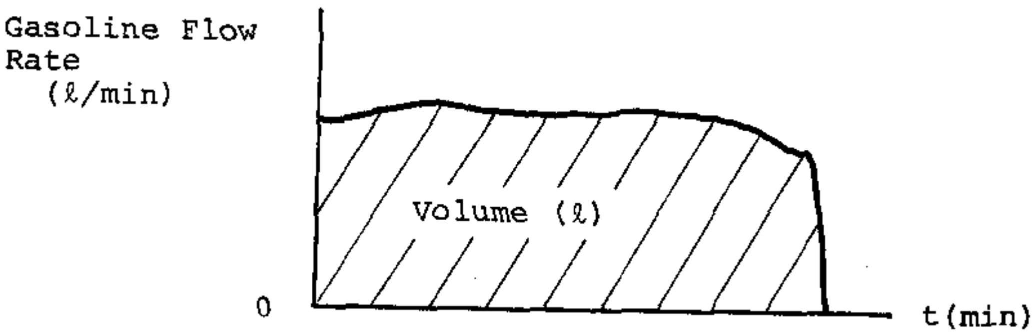


Figure 13

(4) Whereas the speedometer on a car gives the rate of change of distance with time, the odometer integrates this rate to give the total distance, ie, it gives the area under the speed-time curve (see Figure 14).

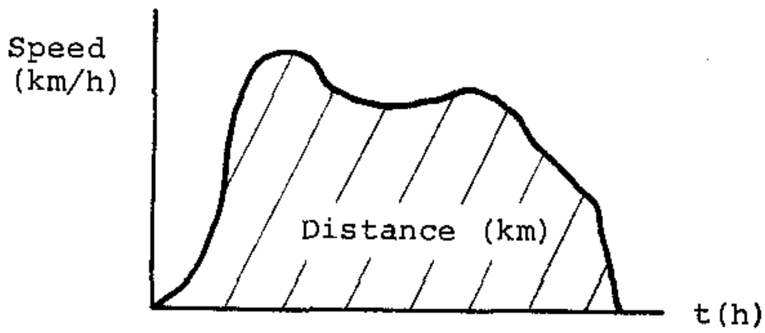


Figure 14

(5) A radiation dosimeter integrates the dose rate to record the total dose received, ie, it gives the area under the dose-rate-time graph (see Figure 15).

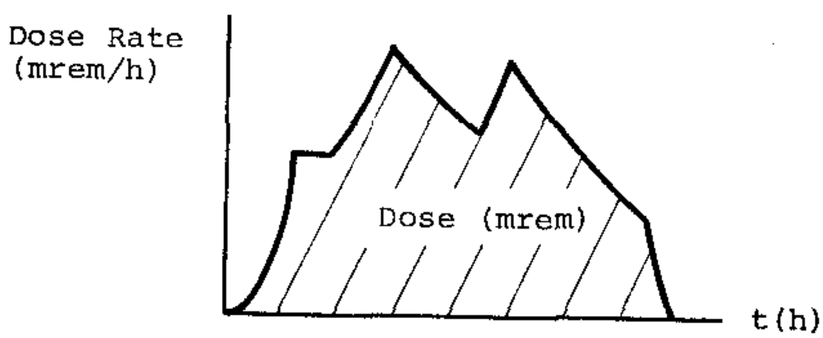
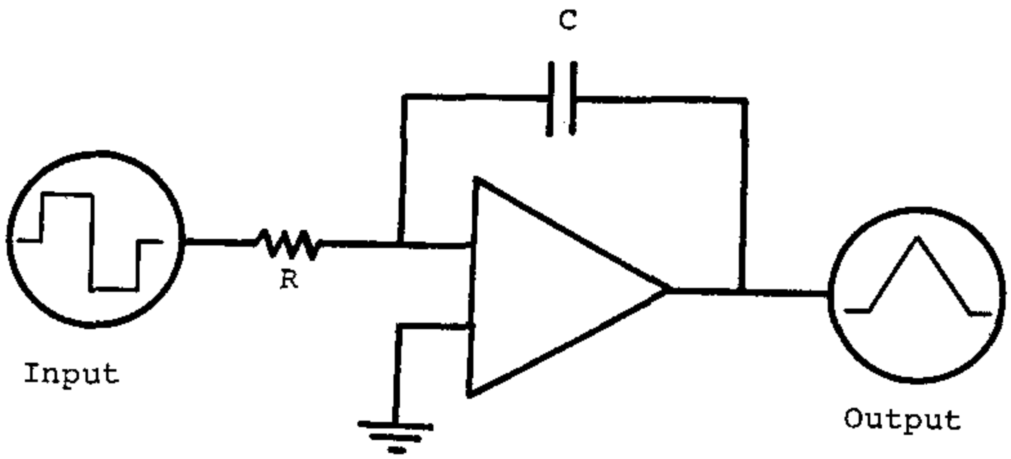


Figure 15

(6) An <u>operational amplifier</u> in the integral mode yields an output which is proportional to the integral of its input (see Figure 16).



<u>Figure 16</u>

VII Integration in Science and Technology

Just as differential calculus provides the notation for writing and techniques for deriving the rate of change of one physical quantity with respect to another, given the one as a function of the other, so integral calculus provides the notation for writing and techniques for deriving a function from its rate of change. Every differential equation listed in Lesson 221.20-5 can be rewritten as an integral,

eg, $v = \frac{ds}{dt} \Rightarrow s = \int v dt$,

$$\frac{dP}{dt} = \frac{\Delta k}{L} P \Rightarrow P = \int \frac{\Delta k}{L} P dt,$$

$$\frac{dV}{dT} = \frac{nR}{P} \Rightarrow V = \int \frac{nR}{P} dT, \text{ etc.}$$

VIII Reset (Integral) Control Mode

Recall (lesson 20-5) that the control signal, $(CS)_P$, from a proportional controller is proportional to the error, e (difference between measured value and set point of controlled parameter):

$(CS)_P = k_p e\ (+b)$, k_p a constant and

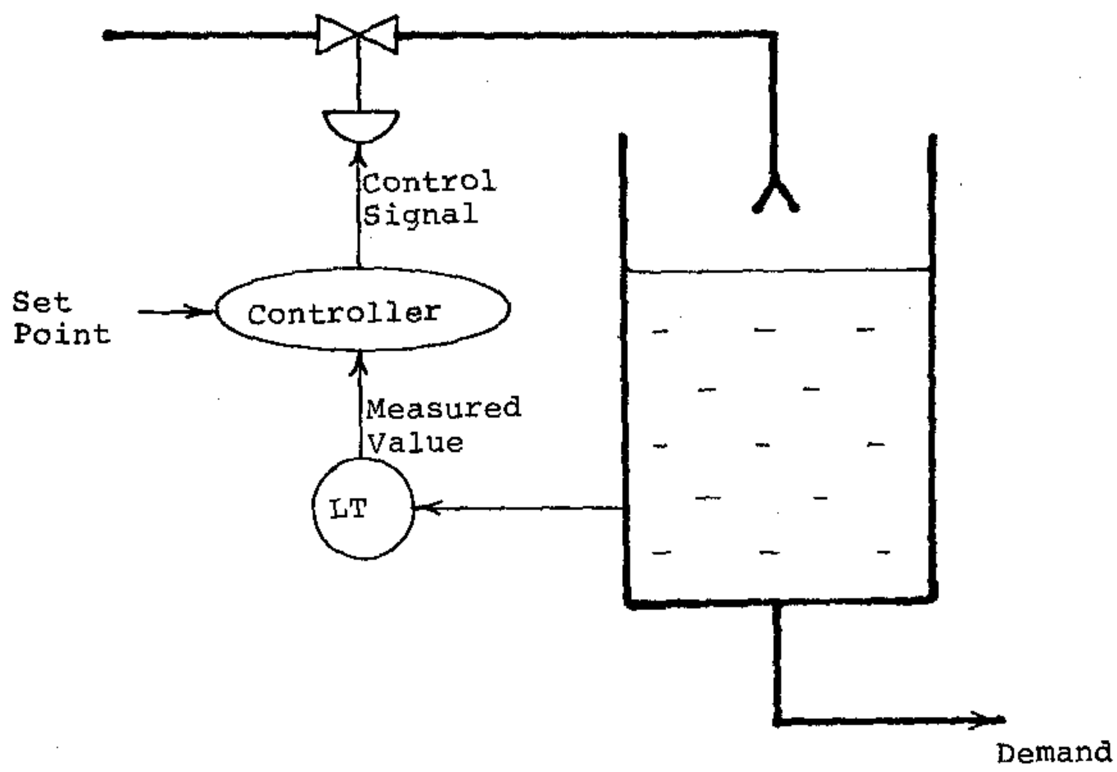
b is the equilibrium bias

One disadvantage of proportional control is *offset* the deviation from set point required merely to generate the required corrective control signal. For example, the tank level in Figure 17 would be offset from set point following any change from equilibrium demand (see lesson 20-5).

This undesirable offset can be eliminated with the use of *reset control*, for which controller output $(CS)_I$ is proportional to the integral of the error signal, e(t):

$(CS)_I = k_I \int edt$, k_I a constant

Reset control drives the error to zero, ie, returns tank level to set point, because $(CS)_I$ keeps changing and varying inflow until e = o. The required integral signal for reset control, $\int edt$, is obtained by passing the error signal through an integrating amplifier.



<u>Figure 17</u> Tank Level Control Loop

In practice, proportional and reset control are usually used together:

$$(CS)_{PI} = k_p e + k_I \int edt \ (+b)$$

Figure 18 illustrates the concept of proportional and integral control signals for two hypothetical fluctuations in the tank level of Figure 17. (NB: assume that demand varies such that the level fluctuates as shown in spite of feedback.) In Figure 18(a) the level drops linearly (from set point) to a new value; in Figure 18(b), the level fluctuates temporarily below set point. The reader should convince himself that the value of the reset component of the control signal at any instant is proportional to the area under the error-time curve up to that instant in both Figures 18(a) and 18(b).

Figure 19 illustrates typical fluctuation in the tank level of Figure 17 for proportional-only control, and for proportional-plus-integral control, following a step increase in demand. Note that offset is eliminated with the introduction of reset control. Similarly, reset control can be used to eliminate offset in controlling reactor power, boiler level, moderator temperature, etc., in a CANDU station.

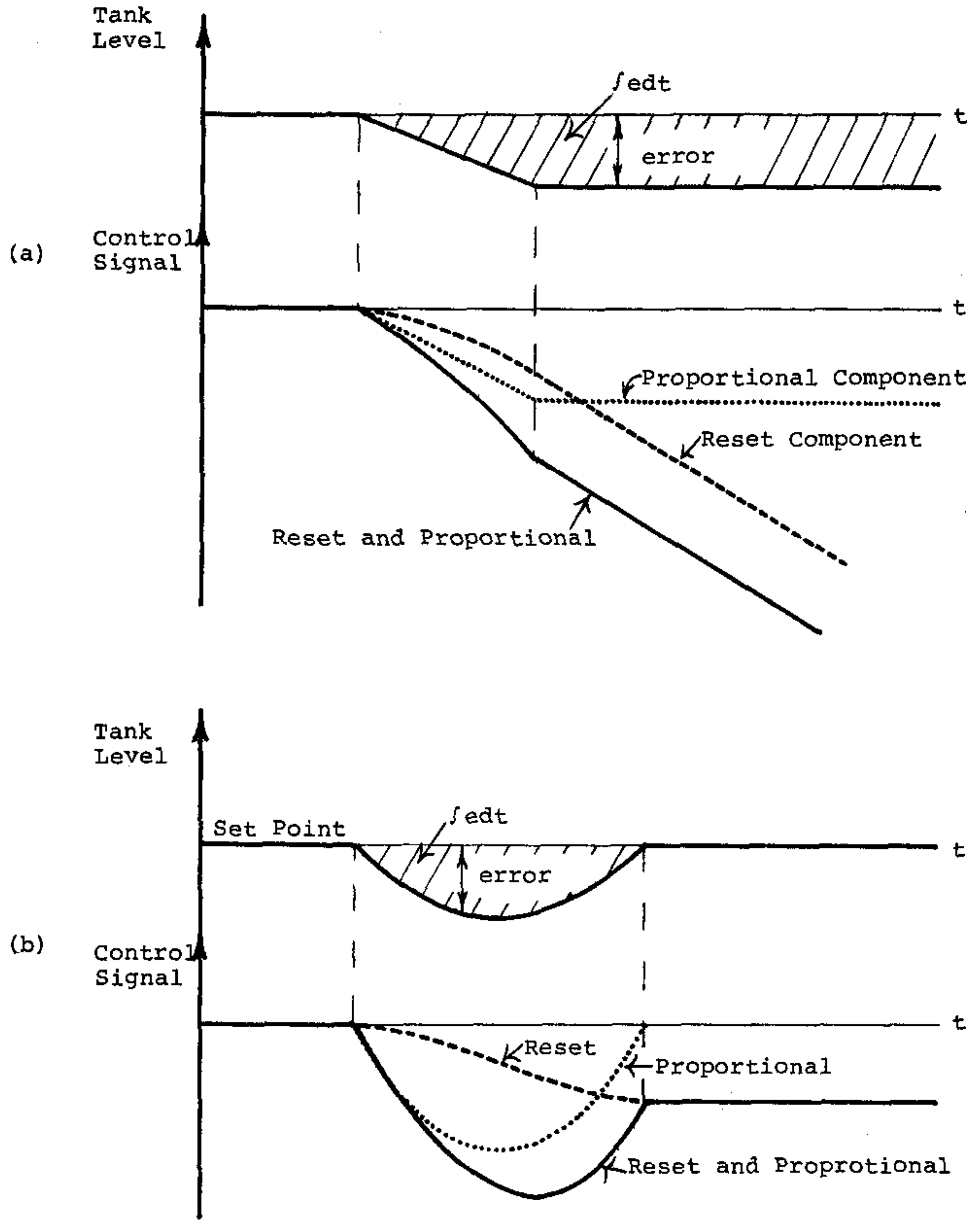
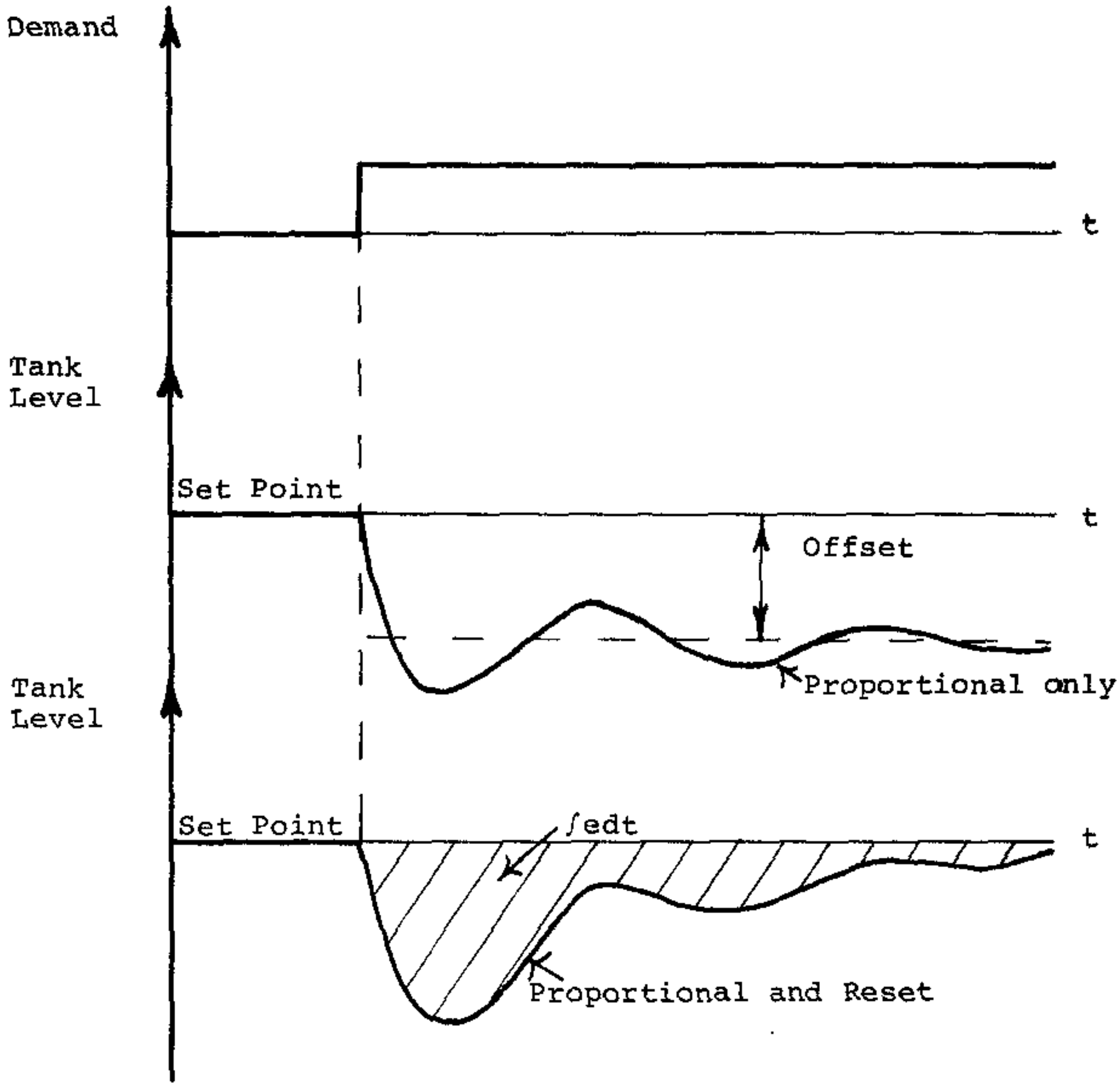


Figure 18 Proportional and Reset Control Signals for Hypothetical Tank Level Fluctuations



<u>Figure 19</u> Tank Level Fluctuations Following Demand Increase

ASSIGNMENT

1. Find the areas bounded by the following. Make large, clear diagrams showing the representative rectangular area increment in each case.

   (a) the x-axis and the curve $y = x^3 - 6x^2 + 8x$

   (b) the parabolas $y = 6x - x^2$ and $y = x^2 - 2x$

   (c) the parabola $y^2 = 4x$ and the line $y = 2x - 4$.

2. The radiation dose rate D(t) mrem/h after t hours is given for a certain work location by the expression

$$D(t) = 600e^{-0.8t}$$

   (a) Plot a graph of D(t) vs t for $0 \leq t \leq 5$

   (b) Calculate (i) the total dose
   (ii) the average dose rate

   received during the first 4 hours.

3. The voltage V(t) after t seconds, across a capacitor of capacitance C farads, as it discharges through a resistor of resistance R ohms, is given by

$$V(t) = V_0 e^{-t/RC},$$

   where V_0 is the capacitor voltage at $t = 0$.

   Recall that electrical power P dissipated in a resistor is given by

$$P = \frac{V^2}{R}$$

   (a) Find a mathematical expression for the average power dissipated in a resistance R which drains a capacitance C for T seconds.

   (b) Suppose a capacitor voltage is restored to 6.0 volts every 2 ms, and the capacitor discharges through a 20 Ω resistor during the 2 ms. If the capacitance is 100 μf, select an appropriate power rating for the resistor from the following:

   $\frac{1}{8}$W, $\frac{1}{4}$W, $\frac{1}{2}$W, 1W, 2W, 5W, 10W.

4. A lead ball is dropped by a parachutist while descending at 6 m/s, from a height of 1000 m above the ground. The acceleration due to gravity is 9.8 m/s^2. Neglecting air resistance, find the

   (a) velocity and displacement functions

   (b) time required for the ball to reach the ground

   (c) average velocity of the ball during its descent

   (d) average height of the ball above ground during its descent

5. Find the amount of work done in stretching a spring from an extension of 0.15 m to an extension of 0.35 m if the spring constant k = 2.4 x 10^2 Nm^{-1}.

6. The boundary of the parabolic dam illustrated below follows the curve

$$y = \frac{x^2}{100},$$

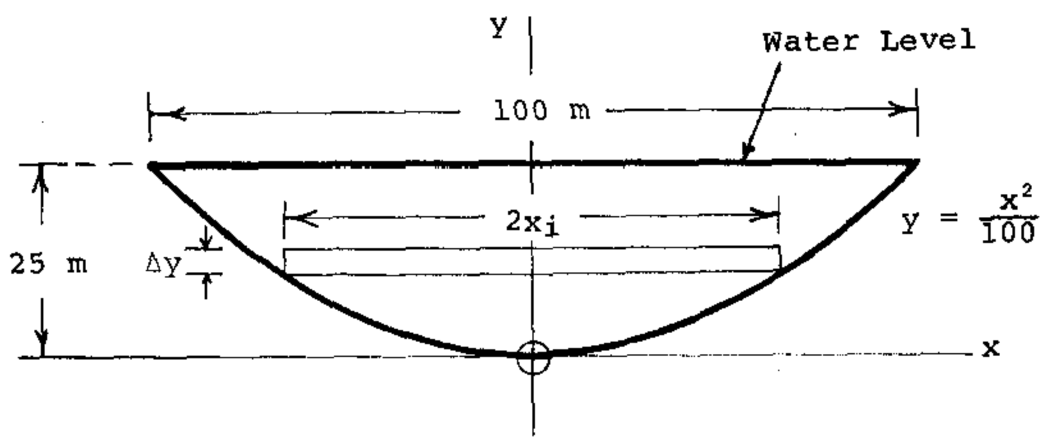
   relative to axes as indicated.

   Find the

   (a) total force exerted against the dam

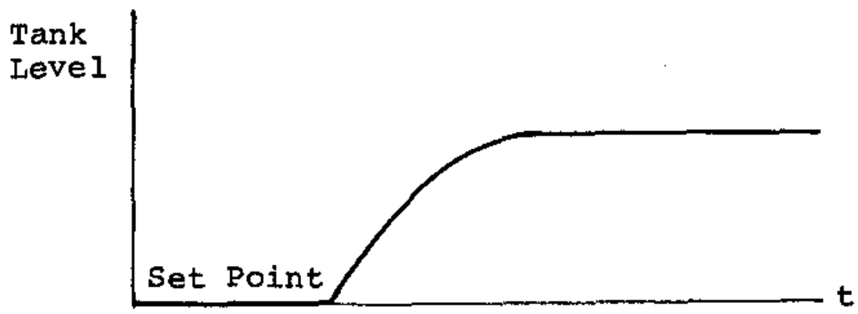
   (b) average pressure exerted against the dam

   (c) center of pressure exerted against the dam



7. The following diagram depicts a hypothetical variation in the tank level of Figure 17. On the same time axis, sketch the following for a proportional-reset controller:

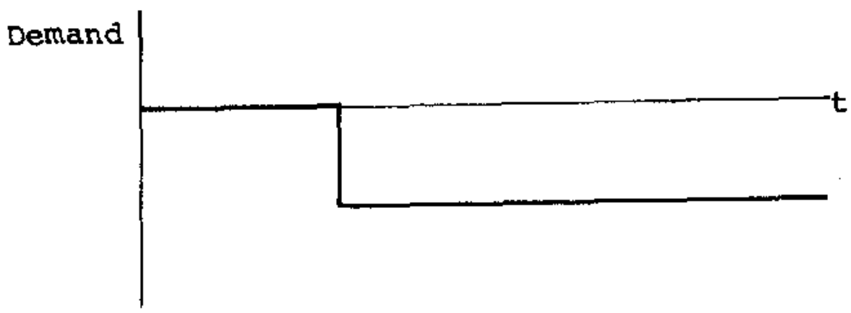
   (a) proportional component of control signal
   (b) reset component of control signal
   (c) total output signal.



8. The following diagram depicts a step decrease in demand flow from the tank of Figure 17. On the same time axis, sketch typical corresponding fluctuations in tank level in the following cases:

   (a) no level control
   (b) proportional only level control
   (c) proportional-reset level control.

   Assume level was at set point prior to demand change. Label the offset in (b).



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